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PROJECTIVE SUBMODEL OF THE OVSYANNIKOV VORTEX

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A submodel of the Ovsyannikov vortex with projective symmetry is studied. Integration of the factor system of the submodel reduces to solving a first-order differential equation which is not solved with respect to the derivative. The properties of the solutions of this equation are studied. It is shown that the submodel describes gas flow with a nonstationary source and a nonstationary sink. The problem of the motion of a gas volume between pistons of cylindrical shapes is studied, and its solution with an invariant shock wave is obtained.

Key words: *Ovsyannikov vortex, partially invariant solutions of the equations of gas dynamics, shock waves.*

Introduction. The term singular vortex refers to the partially invariant solutions of the equations of gas dynamics (EGD) of rank two and defect one constructed on the basis of the rotation group $SO(3)$, which is admitted by the EGD [1]. This class of solutions generalizes the spherically symmetric solutions in the sense that the tangent to the spheres of the velocity component is different from zero. Pokhozhaev suggested that this class of solutions be called the Ovsyannikov vortex. The system of equations of the Ovsyannikov vortex is split into an invariant subsystem that describes the radial motion of gas and equations for a noninvariant function that describes the spherical component of the motion. The latter subsystem is integrated in implicit form over the solutions of the invariant subsystem.

For a complete description of gas flow, it is necessary to find the solutions of the radial subsystem. Using the symmetries admitted by the radial subsystem, it is possible to find its invariant solutions. Chupakhin [2] studied the submodel of gas dynamics generated by the subalgebra $\{so(3), \partial_t\}$, which was called the stationary Ovsyannikov vortex. The homogeneous Ovsyannikov vortex (the submodel generated by the subalgebra $\{so(3), K\}$, where K is a dilatation operator) was investigated in [2, 3]. In these submodels, the invariant functions are expressed in terms of an auxiliary function of one variable and its derivatives. In the case of the stationary Ovsyannikov vortex, this function is a solution of an ordinary differential equation (ODE) of the first order that is not solved with respect to the derivative. For the homogeneous Ovsyannikov vortex, the auxiliary function is a solution of the Schwarz inhomogeneous equation.

In the present paper, we study the projective submodel of the Ovsyannikov vortex. In this submodel, the required functions are also written in terms of an auxiliary function that is a solution of a first-order ODE that is not solved with respect to the derivative. Four integral curves pass through each point of the range of solutions of this equation, which allows one to construct a solution that corresponds to gas flow with a shock wave [4].

1. Ovsyannikov Vortex. Let r, θ, and φ be spherical coordinates, where $0 \leq \theta \leq \pi$, and let $\bar{U}$, V, and W be the radial, latitudinal (in θ), and longitudinal (in φ) velocity components. We introduce the modulus $\bar{H}$ of the tangential velocity component (V, W) and the angle ω of its deviation from the meridian: $V = \bar{H}\cos\omega$ and $W = \bar{H}\sin\omega$.

The representation of the examined partially invariant solution of rank two and defect one with the noninvariant function ω is written as

$$\bar{U} = \bar{U}(t, r), \quad \bar{H} = \bar{H}(t, r), \quad \rho = \rho(t, r), \quad S = S(t, r), \quad \omega = \omega(t, r, \theta, \varphi). \tag{1.1}$$

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The equations of the Ovsyannikov vortex are split into the invariant subsystem describing radial gas motion

$$D_0\bar{U} + \rho^{-1}p_r = r^{-1}\bar{H}^2, \quad D_0(r\bar{H}) = 0, \quad D_0S = 0, \quad p = f(\rho, S); \tag{1.2}$$

$$kD_0h = h^2 + 1, \qquad k = r/\bar{H}, \qquad h = k(\rho^{-1}D_0\rho + r^{-2}(r^2\bar{U})_r), \tag{1.3}$$

and an overdetermined system for the noninvariant function ω, which after the replacement $h = \tan\tau$ is written as

$$\begin{gathered}\omega_\tau + \cos\omega\,\omega_\theta + (\sin\omega/\sin\theta)\omega_\varphi = -\cot\theta\sin\omega,\\ \sin\theta\sin\omega\,\omega_\theta - \cos\omega\,\omega_\varphi = \cos\theta\cos\omega + \tan\tau\sin\theta,\end{gathered} \tag{1.4}$$

and is then integrated in implicit form [1]. In (1.2)–(1.4), $D_0 = \partial_t + \bar{U}\partial_r$ is operator, k and h are auxiliary functions of the variables t and r, and $\tau = \tau(t, r)$ is modified time.

At the initial time $\tau = 0$, if the particle projection onto the unit sphere S_1 is specified by the coordinates x_0, y_0, and z_0 (which correspond to the spherical coordinates $r_0 = 1, \theta_0, \varphi_0$), the spherical trajectory of the particle (the projection of the trajectory onto the sphere S_1) is described by the vector function

$$\boldsymbol{x}(\tau) = (l(\tau)\boldsymbol{x}_0 + \boldsymbol{m}\sin\tau)/\sin\theta_0, \tag{1.5}$$

where $l(\tau) = \cos\tau\sin\theta_0 + \sin\tau\cos\theta_0\cos\omega_0$ and $\boldsymbol{m} = (-y_0\sin\omega_0, x_0\sin\omega_0, -\cos\omega_0)$. The initial distribution $\omega_0(\theta_0, \varphi_0) = \omega(0, \theta_0, \varphi_0)$ of the function ω should have a special form [1]. The choice $\omega_0(\theta_0, \varphi_0) = \pi/2$ ensures the completeness property of the function $\omega_0(\theta_0, \varphi_0)$: it is defined and continuous on the sphere without poles $0 < \theta_0 < \pi$, $0 \le \varphi_0 < 2\pi$. In this case, formula (1.5) for the spherical particle trajectories is simplified:

$$\boldsymbol{x}(\tau) = \boldsymbol{x}_0\cos\tau + \boldsymbol{m}\sin\tau/\sin\theta_0, \qquad \boldsymbol{m} = (-y_0, x_0, 0). \tag{1.6}$$

This motion is rotationally symmetric (i.e., invariant under rotations around the z axis). The particle trajectories are plane curves. At the initial time $\tau = 0$, the particles are on the sphere without poles, and at the time τ, they occupy the spherical zone $\tau < \theta < \pi - \tau$. For the final description of the gas flow, it is necessary to integrate the invariant subsystem (1.2), (1.3).

2. Projective Submodel. For the case of a polytropic gas with the equation of state

$$p = S\rho^{5/3}, \tag{2.1}$$

the equations of gas dynamics admit the 14-dimensional Lie algebra L_{14}. A characteristic of L_{14} is the projective operator

$$\mathcal{P} = t(t\partial_t + x\,\partial_x + y\,\partial_y + z\,\partial_z - u\,\partial_u - v\,\partial_v - w\,\partial_w - 3\rho\,\partial_\rho - 5p\,\partial_p) + x\,\partial_u + y\,\partial_v + z\,\partial_w,$$

which is not admitted by the equations of a polytropic gas if the adiabatic exponent γ is different from 5/3.

The partially invariant solution of the EGD of rank one and defect one constructed on the basis of the symmetry algebra $L = \{so(3), \partial_t + \mathcal{P}\}$ from the optimal system of subalgebras [5], where $so(3)$ is the three-dimensional Lie algebra of the rotation group $SO(3)$, will be called the *projective submodel of the Ovsyannikov vortex* (*the projective Ovsyannikov vortex*). In the coordinates r, θ, φ, $\bar{U}$, and $\bar{H}$, ω, the algebra L has the set of invariants

$$\frac{r}{\sqrt{t^2+1}}, \quad \bar{U}\sqrt{t^2+1} - \frac{rt}{\sqrt{t^2+1}}, \quad \bar{H}\sqrt{t^2+1}, \quad \rho(t^2+1)^{3/2}, \quad p(t^2+1)^{5/2}.$$

The representation of the solution of the submodel has the form

$$\begin{gathered}\bar{U} = \frac{U(\lambda) + \lambda t}{\sqrt{t^2+1}}, \quad \bar{H} = \frac{H(\lambda)}{\sqrt{t^2+1}}, \quad \rho = \frac{R(\lambda)}{(t^2+1)^{3/2}}, \quad p = \frac{P(\lambda)}{(t^2+1)^{5/2}},\\ c^2 = \frac{C^2(\lambda)}{t^2+1}, \quad C^2(\lambda) = \frac{5}{3}SR^{2/3}, \quad S = S(\lambda), \quad \omega = \omega(t, r, \theta, \varphi), \quad \lambda = \frac{r}{\sqrt{t^2+1}}.\end{gathered} \tag{2.2}$$

Here, the sound velocity $c^2 = \partial p/\partial\rho = 5p/(3\rho)$ and the entropy S are defined by the equation of state (2.1).

The equations of the submodel in question can be obtained from the equations of the Ovsyannikov vortex. Indeed, since the quantities $\bar{U}$, $\bar{H}$, S, ρ, and ω from (2.2) are a particular case of (1.1), they satisfy the invariant subsystem (1.2), (1.3) and the auxiliary subsystem (1.4). Since the quantities $\bar{U}$, $\bar{H}$, S, and ρ from (2.2) are not contained in Eqs. (1.4), the form of the representation of solution (2.2) does not influence the integration of the auxiliary subsystem. The invariant subsystem of the projective Ovsyannikov vortex is obtained by substitution of (2.2) into Eqs. (1.2) and (1.3):

$$\frac{1}{2}\,(U^2)' + \frac{P'}{R} = \frac{H^2}{\lambda} - \lambda, \qquad (\lambda H)' = 0, \qquad S' = 0; \tag{2.3}$$

$$\frac{\lambda}{H}\,Uh' = h^2 + 1, \qquad h = \frac{\lambda}{H}\Big(U(\ln R)' + \frac{1}{\lambda^2}\,(\lambda^2 U)'\Big). \tag{2.4}$$

From (2.3) it follows that $H = a_0/\lambda$ and that the gas flow is isentropic $S = S_0 = \text{const}$. Because spherically symmetric solutions are not considered here, the inequality $\bar{H} = a_0\lambda^{-1}(t^2+1)^{-1} > 0$ ($a_0 > 0$) holds. Expressing the function $U(\lambda)$ from the first equation of (2.4), substituting it into the second equation of (2.4), and integrating, we obtain $R(\lambda) = R_0|h'|/\sqrt{h^2+1}$. Thus, all invariant functions are expressed in terms of λ, h, and h':

$$U = a_0\,\frac{h^2+1}{\lambda^2 h'}, \quad H = \frac{a_0}{\lambda}, \quad R = R_0\,\frac{|h'|}{\sqrt{h^2+1}}, \quad S = S_0, \quad C^2 = C_0\,\frac{h'^{2/3}}{(h^2+1)^{1/3}}, \tag{2.5}$$

where $C_0 = (5/3)S_0R_0^{2/3}$.

Using the equation of state (2.1), which is written in terms of the invariants as $P = S_0R^{5/3}$, we calculate the integral

$$\int \frac{P'}{R}\,d\lambda = \frac{5}{2}\,S_0R^{2/3} + \text{const} = \frac{3}{2}\,C^2(\lambda) + \text{const}, \tag{2.6}$$

where the last equality is valid by virtue of (2.2). Then, in view of (2.5) and (2.6), the first equation in (2.3) gives *the invariant Bernoulli integral*

$$U^2 + 3C^2 = -B(\lambda), \qquad B(\lambda) = (\lambda^4 - 2b_0\lambda^2 + a_0^2)/\lambda^2. \tag{2.7}$$

The constraint on the constant b_0 will be determined below. Substituting the expressions of $U(\lambda)$ and $C^2(\lambda)$ from (2.5) into (2.7), we obtain *the key equation* for the function $h(\lambda)$

$$F(h', h, \lambda) \equiv 3C_0h'^{8/3} + B(\lambda)h'^2(h^2+1)^{1/3} + (h^2+1)^{7/3}a_0^2/\lambda^4 = 0. \tag{2.8}$$

This is a first-order ordinary differential equation which is not solved with respect to the derivative.

3. Properties of the Solutions of the Key Equation. The main notions of the theory of implicit differential equations are given below [6].

The criminant of the equation $F(h', h, \lambda) = 0$ is the set *of singular points* of this equation, i.e., the points of the surface $F = 0$ at which $\partial F/\partial h' = 0$. *The discriminant curve* of the equation $F(h', h, \lambda) = 0$ is the projection of the criminant onto the plane (h, λ) that is parallel to the h' axis and is obtained by elimination of h' from the relations

$$F = 0, \qquad \frac{\partial F}{\partial h'} = 0.$$

A point of the criminant is called an *irregular singular point* if the tangent to the surface $F = 0$ at this point coincides with the *contact plane* $dh = p\,d\lambda$, where p is the value of the derivative $dh/d\lambda$ at the point considered. The set of irregular singular points of the equation $F(h', h, \lambda) = 0$ satisfies the system

$$F = 0, \qquad \frac{\partial F}{\partial h'} = 0, \qquad F_\lambda + h'\,\frac{\partial F}{\partial h} = 0. \tag{3.1}$$

The remaining points of the discriminant curve are called *regular singular points*. The irregular singular points can be of the following types: *a folded saddle, a focus, or a node*. They are obtained from an ordinary saddle, a focus, or a node by the folding operation [6].

Property 1. *The key equation* (2.8) *implies that* $B(\lambda) < 0$. *Then from* (2.7)*, we obtain the following constraints on the variable* λ *and the parameters* a_0 *and* b_0:

$$0 < \tilde{\lambda}_1 < \lambda < \tilde{\lambda}_2 \quad \Big(\tilde{\lambda}_{1,2} = \sqrt{b_0 \mp \sqrt{b_0^2 - a_0^2}}\Big), \quad 0 < a_0 < b_0. \tag{3.2}$$

We note that the requirement $B(\lambda) < 0$ is not sufficient for the existence of solutions of (2.8).

Properties 2. 1. *The discriminant curve* $\partial\Omega$ *of key equation* (2.8) *is given by the equation*

$$h^2 = (\lambda B(\lambda)/æ)^4 - 1 \qquad [æ = 4(C_0^3 a_0^2)^{1/4} > 0]. \tag{3.3}$$

It is defined for $\lambda \in [\delta_1, \delta_2]$ [δ_1 *and* δ_2 *are roots of the polynomial* $\lambda^2 B(\lambda) + æ\lambda$], *is continuous, and bounded.*

2. *The upper part*

$$d_+(\lambda) = ((\lambda B(\lambda)/æ)^4 - 1)^{1/2} > 0 \tag{3.4}$$

of the discriminant curve (3.3) *has a unique extremum* (*maximum*) *at the point*

$$\delta_0 = \Big((b_0 + \sqrt{b_0^2 + 3a_0^2}\,)/3\Big)^{1/2}, \tag{3.5}$$

and $d_+(\lambda)$ *increases monotonically on the segment* (δ_1, δ_0) *and decreases monotonically on* (δ_0, δ_2). *The lower part* $d_-(\lambda) = -((\lambda)B(\lambda)/(æ)^4 - 1)^{1/2}$ *of discriminant curve* (3.3) *is symmetric to* $d_+(\lambda)$ *about the* λ *axis.*

Proof. 1. Differentiation of (2.8) with respect to h' yields

$$F_{h'} \equiv 2h'(4C_0 h'^{2/3} + B(\lambda)(h^2 + 1)^{1/3}) = 0.$$

If $h' = 0$, relation (2.8) implies that $a_0 = 0$, which is in contradiction with $a_0 > 0$. Substitution of the solution

$$h'^2 = -B^3(\lambda)(h^2 + 1)/(4^3 C_0^3)$$

into (2.8) leads to Eq. (3.3).

Relation (3.3) implies the inequality

$$(\lambda B(\lambda)/æ)^4 - 1 = æ^{-4}(\lambda^2 B^2(\lambda) + æ^2)(\lambda B(\lambda) - æ)(\lambda B(\lambda) + æ) \geq 0. \tag{3.6}$$

Since $æ > 0$, $\lambda > 0$, and $B(\lambda) < 0$, inequality (3.6) is equivalent to the inequality

$$\lambda B(\lambda) + æ = (\lambda^4 - 2b_0\lambda^2 + æ\lambda + a_0^2)/\lambda \leqslant 0. \tag{3.7}$$

Because $b_0 > 0$ and $æ > 0$ by virtue of (3.2) and (3.3), there are only two changes of sign in the series of coefficients of the fourth-order polynomial. By the Descartes theorem [7], these polynomial has no the roots or has two positive roots δ_1 and δ_2. The set of positive solutions $\lambda > 0$ of Eqs. (3.7) form the segment $[\delta_1, \delta_2]$. Because discriminant curve (3.3) is continuous for $\lambda > 0$, it is bounded on $[\delta_1, \delta_2]$. We note that $d_+(\lambda) > 0$ for $\lambda \in (\delta_1, \delta_2)$.

2. Differentiating (3.4), we obtain

$$d'_+(\lambda) = \frac{1}{2d_+(\lambda)}\,\frac{4\lambda B^3(\lambda)(3\lambda^4 - 2b_0\lambda^2 - a_0^2)}{æ^4}.$$

The polynomial $3\lambda^4 - 2b_0\lambda^2 - a_0^2$ has exactly one positive root δ_0 (3.5), and the polynomial values are negative on the left of δ_0 and positive on the right of it. Because $B(\lambda) < 0$ and $d_+(\lambda) > 0$ on (δ_1, δ_2), it follows that δ_0 is a unique extremum (maximum) of the function $d_+(\lambda)$. Thus, Property 2 is proved.

We note that the following inequalities hold:

$$0 < \tilde{\lambda}_1 < \delta_1 < \delta_2 < \tilde{\lambda}_2. \tag{3.8}$$

Here $\tilde{\lambda}_1$ and $\tilde{\lambda}_2$ are solutions of the equation $B(\lambda) = 0$ and δ_1 and δ_2 are solutions of the equation $\lambda B(\lambda) + æ = 0$. Inequalities (3.8) are valid because the curve $h = \lambda B(\lambda) + æ$ is located above the curve $h = B(\lambda)$.

Property 3. *Solutions of the key equation* (2.8) *exist only in the region* $\Omega = \{(h, \lambda)\colon h^2 \leq (\lambda B(\lambda)/æ)^4 - 1\}$, *which is bounded by the discriminant curve* (3.3). *Four integral curves pass through each point* $(h_0, \lambda_0) \in \Omega$; *two of them describe monotonically increasing solutions and the other two monotonically decreasing solutions. For none of the integral curves do vertical* (*parallel to the* h) *asymptotes exist.*

Proof. Let us find the number of real solutions of Eqs. (2.8). After the replacement $q = h'^{2/3}$, Eq. (2.8) is written using the polynomial $\bar{F}(q, h, \lambda)$ of the fourth order in the variable q:

$$\bar{F}(q, h, \lambda) \equiv 3C_0 q^4 + B(\lambda)(h^2 + 1)^{1/3} q^3 + (h^2 + 1)^{7/3} a_0^2/\lambda^4 = 0. \tag{3.9}$$

The unique extremum of the polynomial $\bar{F}$ is the minimum point $q_{\min} = -B(\lambda)(h^2 + 1)^{1/3}/(4C_0)$. Because in the series of coefficients of the polynomial $\bar{F}$ there are only two changes of sign, it follows from the Descartes theorem [7] that it does not have real roots or has two positive real roots. Since for large λ, polynomial (3.9) takes positive

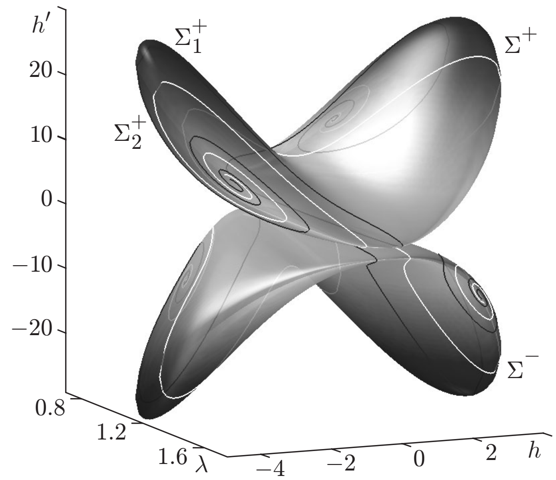

Fig. 1

values, it has two real positive roots $q_1(h, \lambda)$ and $q_2(h, \lambda)$ if and only if the inequality $\bar{F}(q_{\min}, h, \lambda) \leqslant 0$ is satisfied at the minimum point $q_{\min}$, i.e.,

$$\bar{F}(q_{\min}, h, \lambda) \equiv (h^2+1)^{4/3}\Big(\frac{a_0^2}{\lambda^4}\,(h^2+1) - \frac{B^4(\lambda)}{4^4 C_0^3}\Big) \leqslant 0,$$

which is equivalent to the condition $(h, \lambda) \in \Omega$. The equality $q_1(h, \lambda) = q_2(h, \lambda)$ is equivalent to the condition $(h, \lambda) \in \partial\Omega$.

Equation (2.8), which is solved with respect to the derivative, is split into four equations:

$$h_1'(\lambda) = q_1^{3/2}(h_1(\lambda), \lambda), \qquad h_2'(\lambda) = q_2^{3/2}(h_2(\lambda), \lambda); \tag{3.10}$$

$$h_3'(\lambda) = -q_1^{3/2}(h_3(\lambda), \lambda), \qquad h_4'(\lambda) = -q_2^{3/2}(h_4(\lambda), \lambda). \tag{3.11}$$

Since $q_1(h, \lambda)$ and $q_2(h, \lambda)$ are finite for $(h, \lambda) \in \Omega$ and roots of the polynomial, they are continuous in h and λ. Solving Eqs. (3.10) and (3.11) subject to the initial conditions $h_i(\lambda_0) = h_0$, we obtain two monotonically increasing $h = h_1(\lambda)$ and $h = h_2(\lambda)$ and two monotonically decreasing $h = h_3(\lambda)$ and $h = h_4(\lambda)$ solutions of the key equation (2.8). Since $q_1(h, \lambda)$ and $q_2(h, \lambda)$ do not become infinite in the examined region, the integral curves $h_i(\lambda)$ do not have vertical asymptotes. Thus, Property 3 is proved.

Because $q_1(h, \lambda)$ and $q_2(h, \lambda)$ are continuous in Ω and $q_1(h, \lambda) = q_2(h, \lambda)$ only for $(h, \lambda) \in \Omega$, it follows that in the space $\mathbb{R}^3(h', h, \lambda)$ the first and second equations in (3.10) form the lower (Σ_1^+) and upper (Σ_2^+) parts, respectively, of the surface Σ^+, which is located in the upper half-space $h' > 0$. Similarly, Eqs. (3.11) form the surface Σ^-, which is symmetric to Σ^+ about the plane $h = 0$. Figure 1 shows the surfaces Σ^+ and Σ^- in the space $\mathbb{R}^3(h', h, \lambda)$ and some integral curves of Eq. (2.8) with parameters $a_0 = 1$ and $b_0 = 2$ on these surfaces.

Each integral curve $h = h_i(\lambda)$ from Ω corresponds to an integral curve $h = h_i(\lambda), h' = h_i'(\lambda)$ on the surface Σ^+ or Σ^-. The key equation (2.8) admits the reflection $h \to -h$, $h' \to -h'$, and, hence, each integral curve of the surface Σ^- can be obtained from an integral curve of the surface Σ^+ by reflection about the λ axis. Therefore, it suffices to study the properties of the integral curves only on one surface, for example, Σ^+.

Subsequently, the notation $\{h = h(\lambda)\} \subset \Sigma_i^+$ $(i = 1, 2)$ implies that the integral curve $\{h = h(\lambda)\} \subset \Omega$ is a solution of the ith Eq. (3.10); in this case, the curve $h' = h'(\lambda), h = h(\lambda)$ lies on the surface Σ_i^+.

Properties 4. 1. *For the functions $q_1(h, \lambda)$ and $q_2(h, \lambda)$, the equality $q_2(h, \lambda) = q_1(h, \lambda)$ for $(h, \lambda) \in \partial\Omega$ is satisfied and the inequality $q_2(h, \lambda) > q_1(h, \lambda)$ for $(h, \lambda) \in \Omega \setminus \partial\Omega$ is valid.*

2. *Each regular singular point of the discriminant curve $\partial\Omega$ is a branching point* (*a stagnation point*)*: two integral curves — $\{h = h_1(\lambda)\} \subset \Sigma_1^+$ and $\{h = h_2(\lambda)\} \subset \Sigma_2^+$ of the key equation* (2.8) *— leave or enter this point. At these points the integral curves $h = h_1(\lambda)$ and $h = h_2(\lambda)$ have a common tangent.*

3. *At the regular singular points of the discriminant curve $\partial\Omega$, the second derivative $h''(\lambda)$ of the solution of the key equation* (2.8) *becomes infinite.*

Proof. Properties 4.1 and 4.2 directly follow from the properties of the integral curves $h = h_i(\lambda)$ and the functions $q_i(h, \lambda)$ (see Property 3). Property 4.2 can also be obtained as a consequence of the Cibrario theorem [6] on the normal form of an equation that is not solved with respect to the derivative in the neighborhood of a regular singular point.

To prove Property 4.3, we differentiate the equations $F(h'(\lambda), h(\lambda), \lambda) = 0$ with respect to the variable λ and obtain $h'' = -(h'F_h + F_\lambda)/F_{h'}$. The statement follows from the definition of a regular singular point: $F_{h'} = 0$ and $h'F_h + F_\lambda \neq 0$.

Properties 5. 1. *On the surface Σ^+ exactly two irregular singular points exist.*

2. *The integral curves wrap around the surface Σ^+ in the direction from one irregular singular point to another — unwinding around one point and then winding around the other. The irregular singular points are folded focuses* (see Figs. 1 and 2).

Proof. 1. Expressions of the functions h^2 and h' in terms of λ are found from the first two equations in (3.1): $h^2 = (\lambda B(\lambda)/æ)^4 - 1$ and $h' = (-4a_0^2 B^7(\lambda)/æ^8)^{1/2}$. Substitution of them into the third equation of (3.1) yields

$$-\frac{4a_0^2\lambda^{7/3}B^{25/3}}{k^{28/3}}\left[\operatorname{sign}(h)\Big((\lambda B(\lambda)/æ)^4 - 1\Big)^{1/2} a_0\lambda(-B)^{1/2} + (3\lambda^4 - 2b_0\lambda^2 - a_0^2)\right] = 0. \tag{3.12}$$

The factor in front of the square brackets in (3.12) does not vanish. Therefore, the number of solutions (3.12) that lie on the segment $[\delta_1, \delta_2]$ is equal to the number of points of intersection of the discriminant curve (3.3) and the curve

$$h = \chi(\lambda) \equiv -\frac{3\lambda^4 - 2b_0\lambda^2 - a_0^2}{a_0\lambda\sqrt{-B(\lambda)}}.$$

The function $\chi(\lambda)$ vanishes only at the point δ_0 (3.5), at which both parts $d_+(\lambda)$ and $d_-(\lambda)$ of the discriminant curve reach the extremum. Therefore, if one proves that the function $\chi(\lambda)$ decreases monotonically on $[\delta_1, \delta_2]$, the Properties 2.2 imply that the discriminant curve (3.3) and the function $h = \chi(\lambda)$ have exactly two point of intersection.

By virtue of inequalities (3.8), it suffices to prove that $h = \chi(\lambda)$ is monotonic on the interval $(\tilde\lambda_1, \tilde\lambda_2)$, i.e., $\chi'(\lambda)\Big|_{(\tilde\lambda_1,\tilde\lambda_2)} < 0$. The equality

$$\chi'(\lambda) = 2\psi(\lambda)/(a_0\lambda^2(-B(\lambda))^{3/2}),$$

where $\psi(\lambda) = 3\lambda^6 - 9b_0\lambda^4 + 7a_0^2\lambda^2 + 2b_0^2\lambda^2 - 3a_0^2 b_0$, is valid. The equation $\psi'(\lambda) \equiv 2\lambda(9\lambda^4 - 18b_0\lambda^2 + 7a_0^2 + 2b_0^2) = 0$ has only two positive roots: $\mu_{1,2} = (b_0 \pm \sqrt{7(b_0^2 - a_0^2)}/3)^{1/2}$ (μ_1 is the maximum point and μ_2 is the minimum point; the inequalities $\tilde\lambda_1 < \mu_1 < \mu_2 < \tilde\lambda_2$ are satisfied). Constraints (3.2) on the constants a_0 and b_0 imply the inequalities

$$\psi(\tilde\lambda_1) = -4(b_0^2 - a_0^2)(b_0 - \sqrt{b_0^2 - a_0^2}\,) < 0, \qquad \psi(\tilde\lambda_2) = -4(b_0^2 - a_0^2)(b_0 + \sqrt{b_0^2 - a_0^2}\,) < 0.$$

Therefore, $\psi(\lambda) < 0$ and $\chi'(\lambda) < 0$ for $\lambda \in (\tilde\lambda_1, \tilde\lambda_2)$.

2. Let the branching points $\alpha_k \in \partial\Omega$ and the stagnation point $\beta_k \in \partial\Omega$ ($k = 1, 2, \dots$) be chosen in such a manner that the integral curves $\{h = h_{1,k}(\lambda)\} \subset \Sigma_1^+$ and $\{h = h_{2,k}(\lambda)\} \subset \Sigma_2^+$ leave the point α_k and the integral curves $\{h = h_{1,k}(\lambda)\} \subset \Sigma_1^+$ and $\{h = h_{2,k+1}(\lambda)\} \subset \Sigma_2^+$ enter the point β_k (Fig. 2).

From formulas (3.10) and Properties 4.1 and 4.2 it follows that the integral curves $h = h_{1,k}(\lambda)$ and $h = h_{2,k}(\lambda)$ have a common tangent at the points α_k and that and in a neighborhood of the point α_k, the curve $h = h_{1,k}(\lambda)$ is on the right of the curve $h = h_{2,k}(\lambda)$, i.e., $h'_{2,k}(\lambda) > h'_{1,k}(\lambda)$. The integral curves $h = h_{1,k}(\lambda)$ and $h = h_{2,k}(\lambda)$ have no points of intersection except for the point α_k. Indeed, since $h_{1,k}(\lambda)$ and $h_{2,k}(\lambda)$ increase monotonically, the inequality $h'_{2,k}(\lambda_*) < h'_{1,k}(\lambda_*)$ is satisfied at the point of their intersection λ_* (if such exists), which is in contradiction to Properties 4.1. Hence, the curve $h = h_{2,k}(\lambda)$ is located to the left of the curve $h = h_{1,k}(\lambda)$. Similarly, it is proved that the integral curve $h = h_{2,k+1}(\lambda)$ is located to the right of the integral curve $h = h_{1,k}(\lambda)$. Therefore, if the arc $l(\alpha_k, \beta_k)$ of the discriminant curve connects the points α_k, β_k and contains the point β_{k+1},

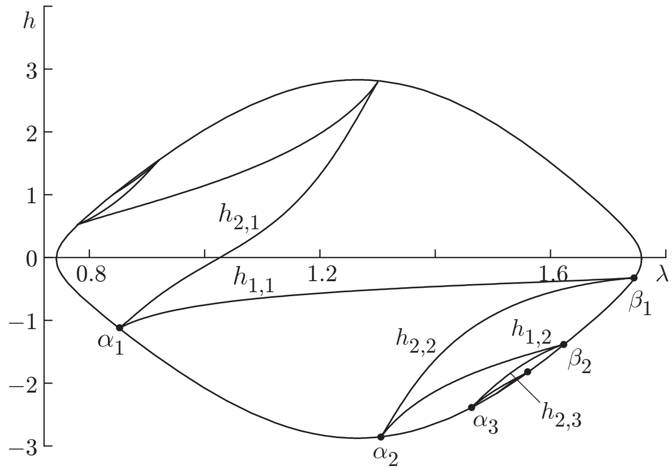

Fig. 2

then $\alpha_{k+1} \in l(\alpha_k, \beta_k)$ and the embedding $l(\alpha_1, \beta_1) \supset l(\alpha_2, \beta_2) \supset \cdots \supset l(\alpha_*, \beta_*)$ is valid. Here $\alpha_* = \lim \alpha_k$ and $\beta_* = \lim \beta_k$ as $k \to \infty$.

We assume that $\alpha_* \neq \beta_*$. Then, the closedness of Ω, the smoothness of the functions $h_{1,k}(\lambda)$ and $h_{2,k}(\lambda)$, and the relative position of their plots imply the convergence: $h_{1,k}(\lambda), h_{2,k}(\lambda) \to g(\lambda)$ and $h'_{1,k}(\lambda), h'_{2,k}(\lambda) \to g'(\lambda)$ as $k \to \infty$. Here $h = g(\lambda)$ is a smooth curve in Ω which connects the points α_* and β_*. According to (3.10), the latter limit is equivalent to the limit $q_1^{3/2}(h_{1,k}(\lambda), \lambda)$, $q_2^{3/2}(h_{2,k}(\lambda), \lambda) \to g'(\lambda)$ as $k \to \infty$, and, consequently, by virtue of the continuity of $q_i(h, \lambda)$, the equality $g'(\lambda) = q_1^{3/2}(g(\lambda), \lambda) = q_2^{3/2}(g(\lambda), \lambda)$ is true. From these equalities and Properties 4.1 it follows that: a) the curve $(g'(\lambda), g(\lambda), \lambda)$ lies on the criminant of the key equation; b) at the points of the curve $(g'(\lambda), g(\lambda), \lambda)$, the tangent to the surface Σ^+ coincides with the contact plane. Therefore, the curve $h = g(\lambda)$ consists of irregular singular points. By virtue of Properties 5.1, this implies that $\alpha_* = \beta_*$.

In the space of variables (h', h, λ), the integral curves $(h'_{1,k}(\lambda), h_{1,k}(\lambda), \lambda) \subset \Sigma_1^+$ and $(h'_{2,k}(\lambda), h_{2,k}(\lambda), \lambda) \subset \Sigma_2^+$ form one curve, which wraps the surface Σ^+ and, approaching the point α_*, performs an infinite number of rotations around it. Such behavior of integral curves near an irregular singular point is characteristic only of a folded focus. The fact that the other irregular singular point of the surface Σ^+ is also a folded focus is established similarly. Thus, Properties 5 are proved.

4. Gas Flow. Let an integral curve $h = h(\lambda)$ intersects the discriminant curve $\partial\Omega$ at the points $\lambda = \lambda_1$ and $\lambda = \lambda_2$ $(\lambda_1 < \lambda_2)$. According to the representation $\lambda = r/\sqrt{t^2+1}$, in the space of events $\mathbb{R}^4(r, \theta, \varphi, t)$, the images of the points λ_1 and λ_2 are the hyperboloids Γ_1 $(r_1 = \lambda_1\sqrt{t^2+1})$ and Γ_2 $(r_2 = \lambda_2\sqrt{t^2+1})$, whose projections in the space $\mathbb{R}^3(r, \theta, \varphi)$ are the spheres with variable radii

$$r = \lambda_1\sqrt{t^2+1} \quad \text{for } S_1(t); \qquad r = \lambda_2\sqrt{t^2+1} \quad \text{for } S_2(t).$$

Below it will be shown that the spheres $S_1(t)$ and $S_2(t)$ are acoustic characteristics.

Statement 1. *The quantities* (2.2) *and* (2.5) *on the solution* $h = h(\lambda)$ *with the function* $\omega(\tau, \theta, \varphi)$ *defined by the initial distribution* $\omega(0, \theta, \varphi) \equiv \pi/2$ *specify the motion of the gas volume between the source* $S_1(t)$ *and the sink* $S_2(t)$. *At the initial time* $t = 0$, *the gas occupies the volume* $\Pi = \{(r_0, \theta_0, \varphi_0)\colon |\tau(r_0)| < \theta_0 < \pi - |\tau(r_0)|,\ \lambda_1 \le r_0 \le \lambda_2\}$ *and its state is defined by*

$$\bar{U}\Big|_{t=0} = a_0\,\frac{h^2+1}{r_0^2 h'}, \qquad \bar{H}\Big|_{t=0} = \frac{a_0}{r_0}, \qquad \rho\Big|_{t=0} = R_0\,\frac{|h'|}{\sqrt{h^2+1}},$$
$$S\Big|_{t=0} = S_0, \qquad \omega\Big|_{t=0} = \omega(\tau(r_0), \theta_0, \varphi_0), \tag{4.1}$$

where $h = h(r_0)$, $h' = h'(r_0)$, $\tau(\lambda) = \arctan h(\lambda)$, *and* $(r_0, \theta_0, \varphi_0) \in \Pi$.

The particles that start at the initial time $t = 0$ *start from the spherical zone* $|\tau(r_0)| < \theta < \pi - |\tau(r_0)|$ *of a sphere of radius* r_0 *move on the trajectories*

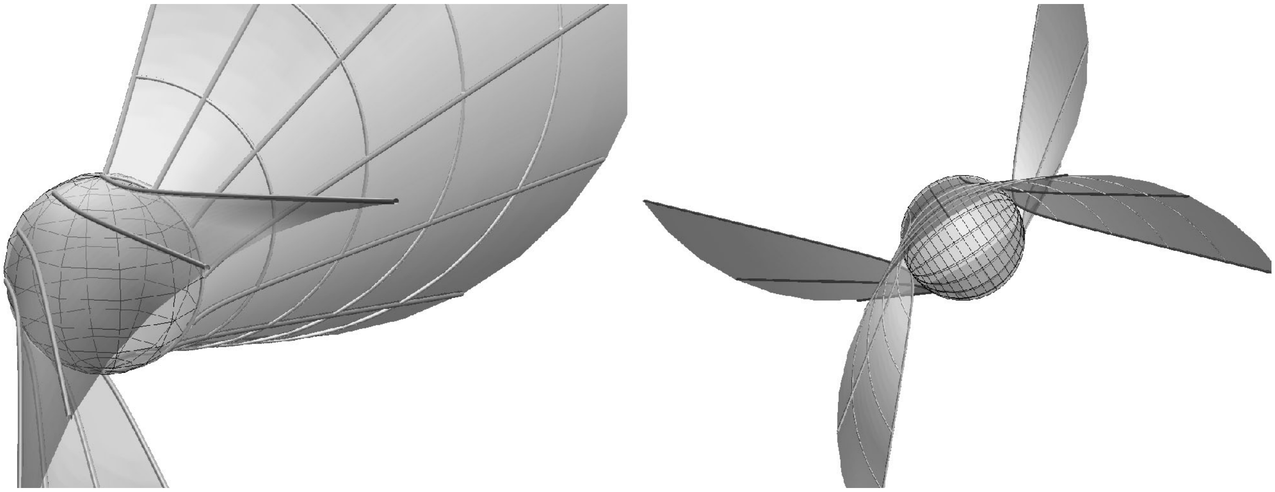

Fig. 3

$$\boldsymbol{x}(t) = r(t)(\boldsymbol{x}_0 \cos\tau(\lambda(t)) + \boldsymbol{m}\sin\tau(\lambda(t))/\sin\theta_0), \tag{4.2}$$

where $r_0(x_0, y_0, z_0) \in \Pi$ *is the initial particle position in Cartesian coordinates,* $\boldsymbol{x}_0 = (x_0, y_0, z_0)$*,* $\boldsymbol{m} = (-y_0, x_0, 0)$*, and at the time* t*, they occupy the spherical zone*

$$|\tau(\lambda(t))| < \theta < \pi - |\tau(\lambda(t))|, \qquad r(t) = \lambda(t)\sqrt{t^2+1}. \tag{4.3}$$

Here $r(t)$ *is the radius of the sphere and* $\lambda = \lambda(t)$ *is a solution of the Cauchy problem*

$$\frac{d\lambda}{dt} = \frac{a_0}{t^2+1}\,\frac{h^2(\lambda)+1}{\lambda^2 h'(\lambda)}, \qquad \lambda(0) = r_0. \tag{4.4}$$

The source and sink are located in the spherical zones S_{f1} *and* S_{f2} *[defined by inequalities* (4.3) *with* $\lambda(t) = \lambda_1, \lambda_2$*] of the spheres* $S_1(t)$ *and* $S_2(t)$*. The flow for them is* $Q(t) = 4\pi R_0 a_0/(t^2+1)$*. At the source and sink, the particle acceleration is infinite and the quantities* (2.2) *are finite.*

Proof. The initial data (4.1) are found from (2.2) and (2.5) at $t = 0$. Formula (4.2) is obtained from the spherical component (1.6) of the motion with allowance for the radial component $r(t)$. Equation (4.4) is obtained by the substitution of $r(t) = \lambda(t)\sqrt{t^2+1}$ into the equations of radial gas motion $dr/dt = \bar{U}(r,t)$, $r(0) = r_0$.

Relation (4.2) implies the formula $z'(t) = z_0 r(t)\cos\tau(\lambda(t))$, which defines the size of the spherical zone (4.3). In particular, at the time t_s at which $\tau(\lambda(t_s)) = 0$, the particles occupy the entire sphere of radius $r(t_s)$ without poles. The region Π is defined at the time $t = 0$ by the spherical zones (4.3), where $\lambda(0) = r_0 \in [\lambda_1, \lambda_2]$.

For an integral curve $\{h = h(\lambda)\} \subset \Omega$, the inequality $h'(\lambda)|_{[\lambda_1^*, \lambda_2^*]} > 0$ is satisfied, and hence, $\bar{U}|_\Pi > 0$ and the source and sink are located on the spheres $S_1(t)$ and $S_2(t)$. The gas flow through the surface $S_i(t)$ $(i = 1, 2)$ is calculated by the formula $Q_i(t) = \int\limits_{S_{fi}} \rho u_i \, dS$ taking into account that the normal velocity component of the gas flow through the surface S_i relative to the velocity of this surface is $u_i = \bar{U}(\lambda_i, t) - d(\lambda_i\sqrt{t^2+1})/dt = U(\lambda_i)/\sqrt{t^2+1}$ and the area of the spherical zone is $S_{fi} = 2\pi h_{si} r_i$, where $h_{si} = 2r_i\cos\tau(\lambda_i) = 2r_i/\sqrt{h^2(\lambda_i)+1}$ and $r_i = \lambda_i\sqrt{t^2+1}$. The quantities ρ and U are defined by formulas (2.2) and (2.5).

The functions $h(\lambda)$ and $h'(\lambda)$ are bounded, and, hence, the quantities (2.2) are finite. Differentiating the velocity components $\bar{U}$ and, $\bar{H}$ from (2.2) with respect to t to find the acceleration and using Property 4.3, we obtain the infinite particle acceleration on the spheres $S_i(t)$. Thus, Statement 1 is proved.

Figure 3 shows the particle trajectories that start from the meridians of the sphere, the surfaces woven from the trajectories, and the images of the meridians. It is evident that the trajectories are not straight lines.

Let us introduce the integral

$$I(\alpha,\beta)=\frac{1}{a_0}\int\limits_{\alpha}^{\beta}\lambda^2\,d\arctan\,h(\lambda).$$

Statement 2. *In the projective Ovsyannikov vortex, let a gas volume* Γ *move between two spherical pistons* $P_1(t)$, $P_2(t)$, *whose radius varies as*

$$r=\mu_1(t)\sqrt{t^2+1}\quad \textit{for } P_1(t);\qquad r=\mu_2(t)\sqrt{t^2+1}\quad \textit{for } P_2(t),$$

where $\mu_i=\mu_i(t)$ *is a solution of the Cauchy problem* (4.4) *with the initial data* $\mu_i(0)=r_i$ *and* $\lambda_1<r_1<r_2<\lambda_2$. *The following versions of motion of the gas volume are possible:*

1. *For* $I(r_2,\lambda_2)<\pi/2$, *the piston* P_2 *reaches the characteristic* t_* *in a finite time* S_2, *after which the solution fails.*

2. *For* $I(r_2,\lambda_2)\ge\pi/2$, *the gas volume* Γ *moves for an unlimited time. With time, the distance between the pistons* P_1 *and* P_2 *and the height of the spherical zone of each piston* P_i *tend to infinity and the angle that forms the spherical zone of the piston* P_i *tends to the finite quantity* $\pi-2|\tau(\mu_i^*)|>0$, *where* $\mu_i^*=\lim\limits_{t\to\infty}\mu_i(t)$.

Proof. 1. The condition $I(r_2,\lambda_2)<\pi/2$ is obtained by integrating Eq. (4.4) with the function $\mu_2(t)$ instead of $\lambda(t)$:

$$\frac{1}{a_0}\int\limits_{r_2}^{\lambda_2}\mu_2^2\,d\arctan\,h(\mu_2)=\int\limits_0^{t_*}d\arctan\,t=\arctan\,t_*<\frac{\pi}{2}.$$

2. Let the piston P_2 do not reach the characteristic S_2 in a finite time. Then, the fact that the function $\mu_2(t)$ increases monotonically and is bounded $[\lambda_1\leqslant\mu_2(t)\leqslant\lambda_2]$ implies the existence of

$$\mu_2^*=\lim_{t\to\infty}\mu_i(t)\leqslant\lambda_2\quad (i=1,2)$$

and the following estimate is valid:

$$\frac{1}{a_0}\int\limits_{r_2}^{\lambda_2}\lambda^2\,d\arctan\,h(\lambda)\geqslant\frac{1}{a_0}\int\limits_{r_2}^{\mu_2^*}\lambda^2\,d\arctan\,h(\lambda)=\int\limits_0^{\infty}d\arctan\,t=\frac{\pi}{2}.$$

The condition $I(r_2,\lambda_2)\geqslant\pi/2$ is obtained.

If $\mu_2^*>\mu_1^*$, the distance between the pistons P_1 and P_2 increases: $(\mu_2(t)-\mu_1(t))\sqrt{t^2+1}\sim(\mu_2^*-\mu_1^*)\sqrt{t^2+1}\to+\infty$ as $t\to\infty$. Let us show that the inequality $r_2>r_1$ implies that $\mu_2^*>\mu_1^*$. Integration of Eqs. (4.4) with the function $\mu_i(t)$ instead of $\lambda(t)$ on the segment $[r_i,\mu_i^*]$ yields two equalities $I(r_i,\mu_i^*)=\pi/2$, $i=1,2$. Since the integrand I is positive, the embedding of the integration intervals $[r_2,\mu_2^*]\subset[r_1,\mu_1^*]$ is impossible and, hence, $r_2>r_1\Rightarrow\mu_2^*>\mu_1^*$.

The angle that forms the spherical zone of the piston P_i for $t\to\infty$ is defined by formula (4.3) with μ_i^* instead of $\lambda(t)$, and the inequality $\tau(\mu_i^*)=\arctan\,\mu_i^*<\pi/2$ holds. The height of this spherical zone is $\mu_i(t)\sqrt{t^2+1}\cos\tau(\mu_i(t))\to+\infty$. Thus, Statement 2 is proved.

In the space $\mathbb{R}^3(r,\theta,\varphi)$, we consider a spherical surface $D(\lambda_0,t)$, $\lambda_0\in[\lambda_1,\lambda_2]$, whose radius varies as $r=\lambda_0\sqrt{t^2+1}$. Then, for the velocity of gas motion on the surface $D(\lambda_0,t)$ relative to the velocity $D_r=\lambda_0 t/\sqrt{t^2+1}$ of this surface, the velocity component that is normal to $D(\lambda_0,t)$ is $v=\bar{U}(\lambda_0,t)-D_r(\lambda_0,t)=U(\lambda_0)/\sqrt{t^2+1}$. At the points $(h_i'(\lambda_0),h_0,\lambda_0)\in\Sigma_i^+$ $(i=1,2)$ let the functions $U(\lambda)$ and $C(\lambda)$ from (2.5) take the values $U_i(\lambda_0)$ and $C_i(\lambda_0)$ and the quantities $v_i=U_i(\lambda_0)/\sqrt{t^2+1}$ and $c_i=C_i(\lambda_0)/\sqrt{t^2+1}$ are defined.

Statement 3. 1. *At the points* $(h_i'(\lambda_0),h_0,\lambda_0)\in\Sigma_i^+\setminus K$, *where* K *is the criminant of* Σ^+, *the inequalities* $v_1^2>c_1^2$ *and* $v_2^2<c_2^2$ *are satisfied. The integral curves* $\{h=h_1(\lambda)\}\subset\Sigma_1^+\setminus K$ *and* $\{h=h_2(\lambda)\}\subset\Sigma_2^+\setminus K$ *specify the "supersonic" and "subsonic" gas flow regimes.*

The words "supersonic" and "subsonic" are taken in inverted commas because the sound velocity c_i is compared to the normal velocity component v_i of gas motion relative to the surface D.

2. *At the points* $(h'(\lambda_0),h_0,\lambda_0)\in K$, *the equality* $v_1^2=v_2^2=c^2$ *holds. The sphere with* $r=\lambda_0\sqrt{t^2+1}$ *is an acoustic characteristic of the EGD on solution* (2.2), (2.5), *in which* $h=h(\lambda)$ *is an integral curve of Eq.* (2.8) *with the initial condition* $h(\lambda_0)=h_0$.

Proof. 1. Let (h_0, λ_0) be a certain point from $\Omega \setminus \partial\Omega$. We show that at the points $(h_i'(\lambda_0), h_0, \lambda_0) \in \Sigma_i^+ \setminus K$ the inequalities $v_1^2 > c_1^2$ and $v_2^2 < c_2^2$ are satisfied, which is equivalent to $U_1^2 > C_1^2$ and $U_2^2 < C_2^2$. Using the representation of solution (2.5), we write the last two inequalities as

$$h_1'(\lambda_0) < q_*^{3/2}, \qquad q_*^{3/2} < h_2'(\lambda_0), \tag{4.5}$$

where $q_*^{3/2} = (a_0^2/(\lambda_0^4 C_0))^{3/8}(h_0^2+1)^{7/8}$. Since $h_i' = q_i^{3/2}$, from (4.5) follows

$$q_1(h_0, \lambda_0) < q_* < q_2(h_0, \lambda_0). \tag{4.6}$$

Since $q_1(h_0, \lambda_0)$ and $q_2(h_0, \lambda_0)$ are roots of the polynomial $\bar{F}(q, h, \lambda)$, by virtue of Properties 3, inequalities (4.6) are equivalent to the inequality $\bar{F}(q_*, h_0, \lambda_0) < 0$. This leads to the inequality

$$h_0^2 < \lambda_0^4 B^4(\lambda_0)/(4^4 C_0^3 a_0^2) - 1,$$

which is satisfied for $(h_0, \lambda_0) \in \Omega \setminus \partial\Omega$.

2. For $(h_0, \lambda_0) \in \partial\Omega$, the equalities $v_1^2 = v_2^2 = c^2$ are equivalent to the equalities $q_1(h_0, \lambda_0) = q_* = q_2(h_0, \lambda_0)$ or $\bar{F}(q_*, h_0, \lambda_0) = 0$. The latter equality is verified by calculations. Thus, Statement 3 is proved.

Numerical calculations of gas motion for various values of the parameters a_0, b_0, and C_0 show that in the "supersonic" regime defined by the integral curves from Σ_1^+, the inequality $I(\lambda_1, \lambda_2) < \pi/2$ is satisfied, and, hence, at any initial positions of the pistons P_1 and P_2, the particles enclosed between them reach the characteristic S_2 in a finite time. The surface S_2 is the noncontinuability surface of the given solution. Next, it will be shown that by means of a shock wave, the "supersonic" regime can be transformed to the "subsonic" regime.

In the "subsonic" regime of gas motion defined by the integral curves from Σ_2^+, both $I(\lambda_1, \lambda_2) < \pi/2$ and $I(\lambda_1, \lambda_2) > \pi/2$ are possible. In the latter case, with an appropriate choice of the initial radii r_1 and r_2 of the pistons P_1 and P_2 the gas volume enclosed between them moves for an unlimited time.

5. Shock Wave. To join the solutions in the projective Ovsyannikov vortex via a shock wave, we use the approach proposed in [4]. An invariant shock wave $D(\lambda_0, t)$ with the front $\lambda = \lambda_0$ is considered. In the physical space $\mathbb{R}^3(r, \theta, \varphi)$, the shock front is a sphere whose radius depends on time: $r = \lambda_0\sqrt{t^2+1}$. The front propagates in the radial direction at velocity $D_r(\lambda_0, t) = \lambda_0 t/\sqrt{t^2+1}$.

It is necessary to choose integral curves $\{h = h_i(\lambda)\} \subset \Sigma_i^+$ $(i = 1, 2)$ of Eq. (2.8) with parameters a_{0i}, b_{0i} and C_{0i} in such a manner that the quantities (2.2) and (2.5) with the functions $h_1(\lambda)$ and $h_2(\lambda)$ define the gas motion ahead of and behind the shock front, respectively. The gas-dynamic quantities (2.2) and (2.5) at the leading edge of the shock wave are denoted by subscript 1 and those at the rear edge by subscript 2. The gas flow velocity relative to the front in the radial direction $v_i = \bar{U}_i(\lambda_0, t) - D_r = U_i(\lambda_0)/(t^2+1)$.

Zemplén's theorem for the normal velocity component of gas motion relative to the shock wave holds is satisfied by virtue of Statement 3. At the shock front, the Rankine–Hugoniot relation should hold [8]; in terms of the invariant values (2.5) these relations are written as

$$R_2 U_2 = R_1 U_1; \tag{5.1}$$

$$P_2 + R_2 U_2^2 = P_1 + R_1 U_1^2; \tag{5.2}$$

$$5P_2/R_2 + U_2^2 = 5P_1/R_1 + U_1^2. \tag{5.3}$$

The velocity components tangential to the front should also be conserved:

$$H_2 = H_1, \qquad \omega_2 = \omega_1. \tag{5.4}$$

In view of (2.5), the first equation of (5.4) at the front $\lambda = \lambda_0$ implies the relation

$$a_{02} = a_{01} = a_0. \tag{5.5}$$

The second equation of (5.4) at the front $\lambda = \lambda_0$ becomes $\omega_2(\tau_2(\lambda_0), \theta, \varphi) = \omega_1(\tau_1(\lambda_0), \theta, \varphi)$, where $\tau_i = \arctan h_i(\lambda_0)$. Since the functions $\omega_i(\tau, \theta, \varphi)$ $(i = 1, 2)$ are solutions of (1.4) with the common initial data $\omega_i(0, \theta, \varphi) \equiv \pi/2$, the equality $\omega_2(\tau, \theta, \varphi) = \omega_1(\tau, \theta, \varphi)$ holds. We confine ourselves to considering the integral curves $h = h_1(\lambda)$ and $h = h_2(\lambda)$ for which $h_1(\lambda_0) = h_2(\lambda_0) = h_0$. Then, $\tau_1(\lambda_0) = \tau_2(\lambda_0) = \arctan h_0$ and the second equation in (5.4) does not give auxiliary constraints on the parameters.

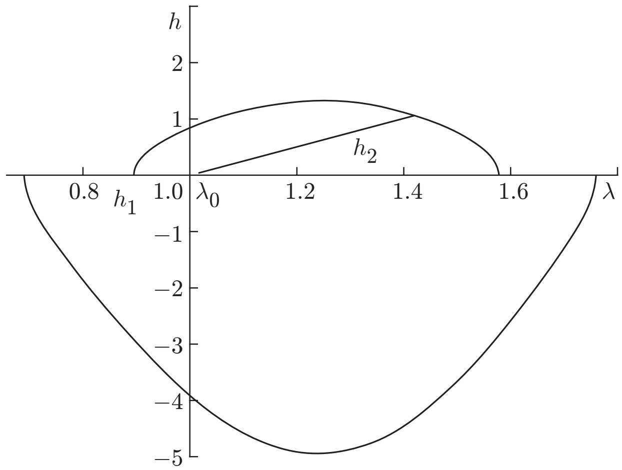

Fig. 4

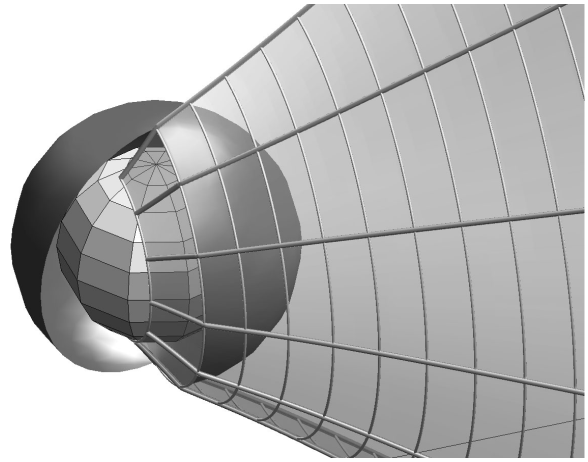

Fig. 5

For a polytropic gas with the equation of state (2.1), the formula $c^2 = (5/3)p/\rho$ holds, which, in view of (2.2), implies $C^2 = (5/3)P/R$. Therefore, Eq. (5.3) is equivalent to $U_2^2 + 3C_2^2 = U_1^2 + 3C_1^2$, whence using the Bernoulli integral (2.7) and equality (5.5), we obtain

$$b_{02} = b_{01} = b_0. \tag{5.6}$$

Equations (5.1) and (2.5) imply $R_{02}\sqrt{h_0^2+1} = R_{01}\sqrt{h_0^2+1}$ and, hence, $R_{02} = R_{01}$.

By virtue of (2.7) and (5.1), equality (5.2) becomes $(U_2-U_1)(4U_1U_2+B(\lambda_0)) = 0$. For shock waves, $U_2 \neq U_1$, and, therefore, by virtue of (2.5) we obtain

$$h_1'(\lambda_0)h_2'(\lambda_0) = \delta, \qquad \delta = -4a_0^2(h_0^2+1)^2/(\lambda_0^4 B(\lambda_0)). \tag{5.7}$$

Statement 4. *In the projective Ovsyannikov vortex, let the gas flow ahead of the shock front* $\lambda = \lambda_0$ *be specified by quantities* (2.5) *with the constants* C_{01}, R_{01}, *and* $S_{01} = (3/5)C_{01}R_{01}^{-2/3}$ *and the integral curve* $\{h = h_1(\lambda)\} \subset \Sigma_1^+$ $[\lambda \leqslant \lambda_0$ *and* $h_1(\lambda_0) = h_0]$ *of the equation*

$$3C_{0i}{h'}_i^{8/3} + k_0{h'}_i^2 + d = 0, \quad k_0 = B(\lambda_0)(h_0^2+1)^{1/3}, \quad d = (h_0^2+1)^{7/3}a_0^2/\lambda_0^4 \tag{5.8}$$

for $i = 1$ *with the parameters* a_0 *and* b_0. *If the integral curve* $\{h = h_2(\lambda)\} \subset \Sigma_2^+$ $[\lambda \geqslant \lambda_0$ *and* $h_2(\lambda_0) = h_0]$ *of Eq.* (5.8) *for* $i = 2$ *with the constant* $C_{02} = -(1/3)(bp_2^2 + d)/p_2^{8/3}$, *where* $p_2 = \delta/h_1'(\lambda_0)$ *and* δ *is defined by the second equality* (5.7), *contains the point* (p_2, h_0, λ_0) *and the inequalities* $p_2 > h_1'(\lambda_0)$ *and* $C_{02} > C_{01}$ *are satisfied, then the gas flow behind the shock wave is defined by the quantities* (2.5) *with the constants* C_{02}, $R_{02} = R_{01}$, *and* $S_{02} = (3/5)C_{02}R_{01}^{-2/3}$ *and the integral curve* $h = h_2(\lambda)$ (see Figs. 4 and 5).

Proof. According to (5.7), the integral curve $h = h_2(\lambda)$ should satisfy the relation $h_2'(\lambda_0) = \delta/h_1'(\lambda_0)$. Substituting $p_2 = \delta/h_1'(\lambda_0)$ into (5.8) for $i = 2$ instead of h_2', we find the expression for C_{02}. For Zemplén's theorem to hold, it is necessary that the gas flow behind the shock front be specified by the integral curve $h = h_2(\lambda)$ with the initial condition $h_2(\lambda_0) = h_0$ which lies on the surface Σ_2^+ of Eq. (5.8) for $i = 2$. The point (p_2, h_0, λ_0) can be on the surface Σ_1^+ or Σ_2^+ of Eq. (5.8) for $i = 2$. For equality (5.7) to hold, it is necessary to verify that $h_2'(\lambda_0) = p_2$, which is equivalent to $(p_2, h_0, \lambda_0) \in \{h = h_2(\lambda)\}$.

By virtue of (2.5), the inequalities $h_2'(\lambda_0) > h_1'(\lambda_0)$, $C_{02} > C_{01}$ guarantee an increase in the density and entropy upon shock-wave propagation. Thus, all necessary conditions at the shock front are satisfied. Statement 4 is proved.

Figure 4 shows the integral curves $h = h_1(\lambda)$ and $h = h_2(\lambda)$ which define the gas flows ahead of and behind the shock front. Figure 5 shows the particle trajectories that start from the spherical zone. At a certain time, a shock wave passes through these gas particles, leading to a change in the trajectories. Because of the choice of the parameter $h_0 = 0$, the shock surface is a sphere without poles.

Let us consider the "supersonic" motion of particles in the gas volume between two pistons P_1 and P_2 (see Statement 2) defined by the integral curve $\{h = h_1(\lambda)\} \subset \Sigma_1^+$ with the initial condition $h_1(r_2) = h_0 = 0$ and a shock wave $D(r_2, t)$ with the front $\lambda = r_2$. Since $\tau(r_2) = \arctan h_0 = 0$, from (4.3) it follows that the shock front is a sphere without poles of radius $r = r_2\sqrt{t^2+1}$. Because $v_1 > 0$, the shock starts moving from the piston P_2 to the piston P_1 at the time $t = 0$. In this case, the gas particles and the shock front move away from the center. Behind the shock wave, the "subsonic" regime is established.

It was noted above that in the projective Ovsyannikov vortex, "supersonic" gas motion exists for a finite time t^*. If $\mu_1(t^*) \geqslant \lambda_2$, the shock wave reaches the piston P_1 earlier than the "supersonic" invariant solution ceases to exist.

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REFERENCES

1. L. V. Ovsyannikov, "Singular vortex," *J. Appl. Mech. Tech. Phys.*, **36**, No. 3, 360–366 (1995).
2. A. P. Chupakhin, "Invariant singular-vortex submodels," *Prikl. Mat. Mekh.*, **67**, No. 3, 390–405 (2003).
3. A. A. Cherevko and A. P. Chupakhin "Homogeneous singular vortex," *J. Appl. Mech. Tech. Phys.*, **45**, No. 2, 209–221 (2004).
4. A. P. Chupakhin, "Self-conjugation of solutions via a shock wave: Limiting shock," *J. Appl. Mech. Tech. Phys.*, **44**, No. 3, 324–335 (2003).
5. A. A. Cherevko, "Optimal system of subalgebras for the Lie algebra of the operators admitted by the system of the equations of gas dynamics with the equation of state $p = f(S)\rho^{5/3}$," Preprint No. 4-96, Institute of Hydrodynamics, Sib. Div., Russian Acad. of Sci., Novosibirsk (1996).
6. V. I. Arnol'd, *Geometrical Methods in the Theory of Ordinary Differential Equations* [in Russian], Izhevsk. Resp. Tip., Izhevsk (2002).
7. A. G. Kurosh, *Course in Higher Algebra* [in Russian], Fizmatgiz, Moscow (1963).
8. L. V. Ovsyannikov, *Lectures on the Fundamentals of Gas Dynamics* [in Russian], Institute of Computational Studies, Moscow–Izhevsk (2003).